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ORDER-TWO CONTINUOUS HAUSDORFF IMAGES OF COMPACT ORDINALS

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ABSTRACT

THEOREM. Let K be a Hausdorff space. The following conditions are equivalent: (a) K is homeomorphic to a compact scattered ordered space; (b) K is an order-two image of a compact ordinal.

§0. Introduction

The theorem stated in the abstract appears as theorem 5 in [4], where it is shown that (a) implies (b). The purpose of this note is to establish it by showing that (b) implies (a). As a T_2-image of a compact scattered T_2-space is again a compact scattered T_2-space ([5], lemma 1), we need the following theorem.

THEOREM 1. *Let K be an order-two T_2-image of a compact well ordered space W. Then K is orderable.*

Let us spell-out this statement. We call a function $f: X \to Y$ an *order-two function* iff for every $y \in Y$ there are at most two solutions in X to the equation $f(x) = y$. A *mapping* means a continuous function. If X, Y are topological spaces and f is a mapping of X onto Y we call Y an *image* of X. If, in addition, Y is a T_2-(Hausdorff) space we call Y a *T_2-image* of X, and if there is an order-two mapping of X onto Y then Y is an *order-two image of* X. A topological space Y is *orderable* iff there is a linear ordering on Y that induces Y's topology.

None of the assumptions of Theorem 1 can be considerably weakened. In [4] it is shown that an order-three T_2-image of a compact well-ordered space (CWOS) need not be orderable. The most familiar nonorderable order-two-T_2-image of a compact ordered space is the circle ($f(t) = e^{2\pi i t}$ is an order-two mapping of the unit interval on the unit circle). The mentioned example in [4] is easily modified to show that even an order-two-T_2-image of a *scattered* compact ordered space

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(SCOS) need not be orderable. Finally compactness cannot be dismissed, as there exist countable T_2-spaces that are not orderable — for example Arens' space ([1], see, e.g. [3], p. 109) — and every such space is a one to one image of the (discrete) ordinal ω.

We outline the argument in §1, and give the detailed proof in subsequent sections. §3 contains a characterization of T_2-order-two images of arbitrary compact T_2-spaces, in terms of properties of their associated reflections (defined there).

A study of the structure of perfect images of CWOSs is found in [6].

§1. Outline of proof — from bottom to roof

Our notation follows [4] and otherwise [3]. In particular, *order* means always linear order; m_A (M_A) denotes the minimal (maximal) member of the ordered set A, whenever it exists. A^* denotes the set A with its order reversed. If B_a is an ordered set for each a in the ordered set A, and $B_a \cap B_{a'} = \emptyset$ for $a \neq a'$, then $\Sigma_a^A B_a$ denotes the ordered set obtained from $\bigcup_{a \in A} B_a$ by retaining the given order on each B_a and requiring $B_a < B_{a'}$ if $a < a'$. A point a in A is called *upper* (*lower*) isolated iff it does not belong to the closure of $\{x \in A : a < x\}$ ($\{x \in A : x < a\}$). a is *semi-isolated* if it is either upper or lower isolated. $\bar{A}$ denotes the order-type of A. The order type of a well-ordered set is the ordinal order-isomorphic to it. A *space* will mean a topological space. An order on a space K is a *consistent order* (with K's topology) if it induces K's topology. Thus, K is orderable iff there is a consistent order on K. Two orders on a set K are *consistent* if they induce the same topology. If K' is an ordered space obtained from an ordered space K by replacing its order by a consistent order, we call K' a *reordering* of K.

Let φ be a mapping of W onto K. We say that an order $<$ on K *respects* φ iff $<$ is a consistent order on K, and $\varphi(w)$ is semi-isolated whenever $\varphi^{-1}(\varphi(w)) = \{w\}$. A mapping φ of W is called *respectable* iff there is an order on $\varphi(W)$ that respects φ. We say that *W is respectable* if every order-two mapping φ of W onto a T_2-space is respectable. Theorem 1 is a consequence of

THEOREM 2. *Every* CWOS *is respectable.*

Theorem 2 is proved by induction on the order type of the CWOS W. It is enough to consider CWOS of type $\omega^\nu \cdot m + 1$ (ν an ordinal, $m \in \omega$) as every CWOS W is homeomorphic to a CWOS of this type ([2], lemma 3). Theorem 3 carries the induction from $\omega^\nu + 1$ to $\omega^\nu \cdot m + 1 = (\omega^\nu + 1) \cdot m$:

THEOREM 3. *Let K_i be a scattered compact T_2-space, $i = 0, 1$, and let K be the topological sum of K_0 and K_1. If K_0 and K_1 are respectable, so is K.*

The proof is given in §4.

COROLLARY. *Let W_i be a scattered compact ordered space* (SCOS) *for $i < m$. If W_i is respectable, $i < m$, so is $W = \Sigma_i^m W_i$.*

Theorem 2 follows from this corollary and

THEOREM 4. *Let W be a* CWOS, $\bar{W} = \omega^\nu + 1$. *If every* CWOS *V with $\bar{V} < \omega^\nu + 1$ is respectable, so is W.*

Our proof of Theorem 4 relies on two Lemmata, whose proofs are given in §2.

LEMMA 1. *Let W be a* CWOS, $\bar{W} = \omega^\nu + 1$, *let* $\rho = \mathrm{cf}(\omega^\nu)$ *and let φ be a mapping of W onto a T_2-space satisfying $\varphi^{-1}(\varphi(M_w)) = \{M_w\}$. Then there is a reordering W' of W such that $\bar{W}' = \omega^\nu + 1$, and $W' = (\Sigma_\alpha^\rho W_\alpha) + \{M_w\}$, where W_α is closed in W', and $\varphi^{-1}(\varphi(W_\alpha)) = W_\alpha$ $(\alpha < \rho)$.*

LEMMA 2. *Let W be a* CWOS *such that every* CWOS *V with $\bar{V} < \bar{W}$ is respectable. Let $W = (\Sigma_\alpha^\rho W_\alpha) + \{M_w\}$, where ρ is a limit ordinal, and W_α is closed in W. Let φ be an order-two-mapping of W such that $\varphi^{-1}(\varphi(W_\alpha)) = W_\alpha$ for $\alpha < \rho$ (hence also $\varphi^{-1}(\varphi(M_w)) = \{M_w\}$). Then φ is respectable.*

PROOF OF THEOREM 4. Let φ be an order-two mapping of W onto a T_2-space. We have to show that φ is respectable. If $\varphi^{-1}(\varphi(M_w)) = \{M_w\}$ then φ is respectable by Lemmata 1 and 2. If not, then for some $w < M_w$ we have $\varphi^{-1}(\varphi(M_w)) = \{w, M_w\}$ as φ is an order-two mapping. Let $W_0 = [m_w, w]$, $W_1 = (w, M_w]$. Then $W = W_0 + W_1$, where W_0, W_1 are CWOSs, and $\bar{W}_0 < \bar{W}$. Let $\varphi_i = \varphi \mid W_i$. Then φ_i is an order-two mapping of W_i onto a T_2-space, $i = 0, 1$. By hypothesis, φ_0 is respectable. Now $\bar{W}_1 = \omega^\nu + 1$ and $\varphi_1^{-1}(\varphi_1(M_{w_1})) = \{M_w\} = \{M_{w_1}\}$; hence, by the previous case φ_1 is also respectable. Hence, by the proof of Theorem 3 (§4), φ is respectable.

§2. Proof of the Lemmata (Milma'la 'ad Lemata)

Let K be an ordered set. A subset V of K is called *convex subset* or *interval* if $a, b \in V$ and $a < t < b$ implies $t \in V$, i.e., $(a, b) \subseteq V$. For arbitrary $U \subseteq K$, $a, b \in U$, let $a \sim b$ iff $(a, b) \subseteq U$. Then $\sim$ is an equivalence relation on U, whose equivalence classes are called the *convex components* of U. If U is open (closed) then every convex component of U is open (closed). If K is compact and U is clopen (= closed and open), then by compactness U has finitely many convex components, each of which is a clopen interval in K. Thus we have

PROPOSITION 2.0. *Let K be a compact ordered space, and let U be a subset of K. Then U is clopen iff U is a finite union of disjointed clopen intervals of K.*

PROPOSITION 2.1. *Let K be a compact ordered space, and let $<$ be its order. Let ρ be an ordinal, and let Λ denote the set of nonzero limit ordinals not exceeding ρ. Let $\{K_\alpha : 0 \leqq \alpha \leqq \rho\}$ be a partition of K into nonempty closed subsets. Let $m_\alpha = m_{K_\alpha}$, $M_\alpha = M_{K_\alpha}$. Assume:*

(i) *K_α is clopen in $[m_\alpha, M_K]$, and for $\alpha \notin \Lambda$, K_α is clopen in K.*

(ii) *Let $\alpha \in \Lambda$. Then for every $\gamma < \alpha$ there is a $k_0 < m_\alpha$ such that $(k_0, m_\alpha) \subseteq \bigcup_{\gamma \leqq \beta < \alpha} K_\beta$.*

Let K' denote the set K ordered by the order relation $<'$ defined by the requirements:

(1) $<' \mid K_\alpha = < \mid K_\alpha$ $(0 \leqq \alpha \leqq \rho)$,

(2) $K' = (\Sigma_\alpha^\rho K_\alpha) + K_\rho$.

Then K' is a reordering of K.

PROOF. We need to show that the identity mapping of K onto K' is a homeomorphism. It is one to one and onto, so since K is compact and K' is Hausdorff, we only have to show continuity. Let $k \in K_\alpha$, and let V' be a K'-open-interval containing k. We shall show that there is a K-open-interval V containing k such that $V \subseteq V'$. Let U be the K-convex-component of K_α containing k. By (i) and Proposition 2.0, U is a clopen interval of the K-closed interval $[m_\alpha, M_K]$ and by (1), (2) it is also a K'-closed interval, clopen in the K'-closed interval K_α. Also, $<' \mid U = < \mid U$ by (1). We now distinguish two cases:

Case 0. $\alpha \notin \Lambda$. Then U is a K-open-interval and also a K'-open-interval. Let $V = V' \cap U$. Obviously, $k \in V$. Since V' is also a K'-open-interval, so is V. Thus V is a U-open-interval, hence a K-open-interval.

Case 1. $\alpha \in \Lambda$. If $m_\alpha < k$, let $V = (U \setminus \{m_\alpha\}) \cap V'$, and repeat the previous argument (with $U \setminus \{m_\alpha\}$ for U) to show that V is a K-open interval containing k. Assume $k = m_\alpha$. Let $V_0 = U \cap V'$. Then $k \in V_0$, and V_0 is a U-open-initial-segment contained in V'. Now V' is an open K'-interval, $m_\alpha \in V'$, and $\alpha \in \Lambda$, so there is by (2) a $\gamma < \alpha$ such that $\bigcup_{\gamma \leqq \beta < \alpha} K_\beta \subseteq V'$. Hence by (ii) there is a $k_0 < m_\alpha$ such that the K-open interval (k_0, m_α) is included in $\bigcup_{\gamma \leqq \beta < \alpha} K_\beta$, hence in V'. Thus, $V = (k_0, m_\alpha) \cup V_0$ is a K-open-interval containing m_α and included in V'.

PROOF OF LEMMA 1. Let $m = m_W$, $M = M_W$, $L = \varphi(W)$, $l = \varphi(M)$. Then L is T_2, compact and scattered, hence zero dimensional [7, p. 168]. Thus, its clopen subsets form a base to the topology. Hence, whenever $L_0 \subseteq L$ is closed and $l \notin L_0$, there is a clopen $L' \subseteq L$ such that $L_0 \subseteq L' \subseteq L \setminus \{l\}$. We shall use this fact to obtain a partition $\{W_\alpha : 0 \leqq \alpha \leqq \rho\}$ of W into closed sets satisfying (i) and (ii) of Proposition 2.1, and also $\varphi^{-1}(\varphi(W_\alpha)) = W_\alpha$.

Let $(a_\alpha)_{\alpha<\rho}$ be an increasing cofinal sequence in $[m, M)$, and let Λ denote the set of nonzero limit ordinals not greater than ρ. We define W_α so that setting $m_\alpha = m_{w_\alpha}$, $M_\alpha = M_{w_\alpha}$, $V_\alpha = \bigcup_{\beta<\alpha} W_\beta$, the following conditions hold:

(a) W_α is a nonempty clopen subset of $[m_\alpha, M]$, W_α is a nonempty clopen subset of W for $\alpha \notin \Lambda$, and $W_\alpha \cap W_\beta = \emptyset$ for $0 \leqq \alpha < \beta \leqq \rho$.

(b) $[m, \max(M_\alpha, a_\alpha)] \subseteq V_{\alpha+1} \subseteq [m, M_{\alpha+1}]$, and $M_\alpha < M$ $(\alpha < \rho)$.

(c) $V_\alpha = [m, m_\alpha)$ for $\alpha \in \Lambda$.

(d) $\varphi^{-1}(\varphi(W_\alpha)) = W_\alpha$ $(0 \leqq \alpha \leqq \rho)$.

(a) and (b) imply that $W_\rho = \{M\}$ and that $\{W_\alpha : 0 \leqq \alpha \leqq \rho\}$ is a partition of W into closed sets. (a) guarantees (i) of Proposition 2.1; (b) and (c) guarantee (ii) (if $\gamma < \alpha \in \Lambda$ then $(M_{\gamma+1}, m_\alpha) \subseteq V_\alpha \setminus V_{\gamma+1} = \bigcup_{\gamma \leqq \beta < \alpha} W_\beta$).

We now turn to the inductive definition of W_α, V_α. By definition, $V_0 = \emptyset$. Assuming W_β defined for all $\beta < \alpha$ so that (a), (b), (c), (d) hold, we note that V_α is bounded in $[m, M)$ (if $\alpha \notin \Lambda$ by (b); if $\alpha \in \Lambda$ by $\alpha < \rho = \mathrm{cf}(M)$). Let $\gamma < \rho$ be smallest such that $V_\alpha < a_\gamma$. Since $\varphi([m, a_\gamma])$ is closed in L and does not contain l, we may choose L' to be a clopen subset of L satisfying $\varphi([m, a_\gamma]) \subseteq L' \subseteq L \setminus \{l\}$. Let $V_{\alpha+1} = \varphi^{-1}(L')$ and let $W_\alpha = V_{\alpha+1} \setminus V_\alpha$. Since $\varphi^{-1}(l) = \{M\}$, $\varphi^{-1}(L')$ is bounded in $[m, M)$ whenever L' is a closed subset of L and $l \notin L'$. Hence $V_{\alpha+1}$ is a clopen subset of W, bounded in $[m, M)$. The verification of (a), (b), (c), (d) is straightforward.

It is left to show that $\bar{W}' = \omega^\nu + 1$. Now $\bar{W}'$ is a CWOS, whose ν'th derived set is nonempty, as W' is homeomorphic to W. Thus, $\omega^\nu + 1 \leqq \bar{W}'$ ([2], lemma 1). On the other hand, it is easily verified by induction that every initial segment of $W' \setminus \{M\}$ has order-type smaller than ω^ν, whence $\bar{W}' = \omega^\nu + 1$.

Proof of Lemma 2. Let $L = \varphi(W)$, $L_\alpha = \varphi(W_\alpha)$, $\varphi_\alpha = \varphi \mid W_\alpha$, $m_\alpha = m_{w_\alpha}$, $M_\alpha = M_{w_\alpha}$ $(\alpha < \rho)$. Let $l = \varphi(M_w)$. Let Λ be the set of nonzero limit ordinals not greater than ρ. Since W_α is a CWOS and $\bar{W}_\alpha < \bar{W}$, W_α is respectable for $\alpha < \rho$, and so since φ_α is an order-two mapping of W_α onto L_α, L_α carries an order $<_\alpha$ that respects φ_α. By $\varphi^{-1}(L_\alpha) = W_\alpha$, we have $L_\alpha \cap L_\beta = \emptyset$ for $\alpha \neq \beta$. Thus, the ordering $<$ on L defined by the requirements:

(1) $< \mid L_\alpha = <_\alpha \mid L_\alpha$ $(\alpha < \rho)$,

(2) $L = (\Sigma_\alpha^\rho L_\alpha) + \{l\}$,

respects φ, provided that it is a consistent ordering of L. Since φ is a closed function, it is a quotient mapping ([3], p. 83–85). Hence $<$ is a consistent ordering of L iff φ is continuous as a function from W into L with the order topology. This in turn holds iff for every transfinite sequence $(a_\alpha)_{\alpha<\tau}$ in W convergent to $a \in W$, $(\varphi(a_\alpha))_{\alpha<\tau}$ is convergent in the ordered space L to $\varphi(a)$. It follows that $<$ is a consistent ordering of L iff for every $\alpha \in \Lambda$, $\varphi(m_\alpha)$ is the $<_\alpha$-first-element of L_α. Hence to complete the proof it is enough to prove

CLAIM. *There is an ordering $<_\alpha$ of L_α that respects φ_α, such that $\varphi(m_\alpha)$ is the $<_\alpha$-first-element of L_α.*

PROOF OF CLAIM. Notice first that whenever K is an ordered space with a maximal element and a minimal element, and $k \in K$ is semi-isolated, there is a reordering K' of K where k is minimal, such that a point of K is semi-isolated in K iff it is semi-isolated in K'. Indeed, if k is semi-isolated, then $K = K_0 + K_1$ where K_0 has a maximal element (or is empty), K_1 has a minimal element (or is empty) and $k = M_{K_0}$ or $k = m_{K_1}$. In the first case let $K' = K_0^* + K_1$, and in the second let $K' = K_1 + K_0$.

Let $l_\alpha = \varphi(m_\alpha) = \varphi_\alpha(m_\alpha)$. We conclude by showing that L_α has an ordering that respects φ_α such that l_α is semi-isolated. Consider two cases.

Case 0. $|\varphi^{-1}(l_\alpha)| = 1$. As $m_\alpha \in \varphi^{-1}(l_\alpha)$, we have in this case $\varphi^{-1}(l_\alpha) = \{m_\alpha\}$. Since m_α is isolated in W_α, and since φ_α maps for each ordinal ν the ν'th derivative $W_\alpha^{(\nu)}$ of W_α onto $L_\alpha^{(\nu)}$ ([5], lemma 1), l_α is isolated in L_α.

Case 1. $|\varphi^{-1}(l_\alpha)| > 1$. Since φ is order two, $|\varphi^{-1}(l_\alpha)| = 2$ and so for some $w \neq m_\alpha$, $\varphi^{-1}(l_\alpha) = \{m_\alpha, w\}$. Since $\varphi^{-1}(L_\alpha) = W_\alpha$, we have $w \in W_\alpha$. Now let $V = W_\alpha \setminus \{m_\alpha\}$. Since $\bar{V} < \bar{W}$, V is respectable. Now $\tilde{\varphi} = \varphi \mid V$ is an order-two mapping of V onto L_α, and $\tilde{\varphi}^{-1}(l_\alpha) = \{w\}$. Hence, there is an ordering $<_\alpha$ of L_α that respects $\tilde{\varphi}$ such that l_α is semi-isolated. Obviously, $<_\alpha$ respects also φ.

This completes the proof of the claim, and of Lemma 2.

§3. Some reflections on T_2-reflections

By a *reflection* of a set K we & $S = \langle a_\sigma, \sigma \in \Sigma\rangle$ mean a permutation ψ of K satisfying $\psi = \psi^{-1}$. Let φ be an order-two function defined on K. Define the *reflection ψ_φ associated with φ* by the requirement that the orbits of ψ_φ are the φ-inverse-images of points in $\varphi(K)$; that is, by the requirement:

$$\varphi^{-1}(\varphi(a)) = \{a, \psi(a)\} \qquad (a \in K).$$

We obviously have $\varphi = \varphi\psi$, as $\varphi(a) = \varphi(\psi(a))$ $(a \in K)$.

A reflection ψ of a space K is called a *T_2-reflection* iff $\psi = \psi_\varphi$ for some order-two mapping φ of K onto a T_2-space. If K is compact, every mapping of K

into a T_2-space is closed, hence a quotient mapping ([3], p. 83–86). Thus, $\varphi(K)$ is homeomorphic with the quotient space $K/\sim$, where $a \sim b$ iff $\varphi(a) = \varphi(b)$. Hence the order-two T_2-images of a compact space K are up to homeomorphism the quotient spaces $K/\sim_\psi$, where ψ varies on all T_2-reflections on K, and $a \sim_\psi b$ iff $a = b$ or $a = \psi(b)$. Let $\mathrm{Cl}(A)$ denote the closure of $A \subseteq K$ in K. The following theorem characterizes the T_2-reflections of a compact T_2-space K:

THEOREM 5. *Let K be a compact Hausdorff space and let ψ be a reflection of K. Then the following are equivalent:*

(1) *ψ is a T_2-reflection.*

(2) *Whenever $A \subseteq K$ and $\mathrm{Cl}(A) \cap \mathrm{Cl}(\psi(A)) = \emptyset$, $\psi \mid \mathrm{Cl}(A)$ is a homeomorphism of $\mathrm{Cl}(A)$ onto $\mathrm{Cl}(\psi(A))$.*

(3) *If A and B are disjointed closed subsets of K, D is dense in A and $\psi(D)$ is dense in B, then $\psi(A) = B$.*

The equivalence of (2) and (3) is safely left to the reader; we prove the equivalence of (1) and (2).

(1) $\Rightarrow$ (2). Let φ be an order-two mapping of K onto a T_2-space satisfying $\psi_\varphi = \psi$. We show first that $\psi(\mathrm{Cl}(A)) = \mathrm{Cl}(\psi(A))$. Let $a \in \mathrm{Cl}(A)$. Let $S = \langle (a_\sigma), \sigma \in \Sigma \rangle$ be a net in A converging to a. Since φ is continuous, $\varphi S = \langle \varphi(a_\sigma), \sigma \in \Sigma \rangle$ converges to $\varphi(a)$. φS has no other limit, since $\varphi(K)$ is T_2 ([3], p. 55–56). Since K is compact, the net $T = \psi S = \langle \psi(a_\sigma), \sigma \in \Sigma \rangle$ has a finer net T' that converges to some $b \in K$. Since T' is a net in $\psi(A)$, b belongs to $\mathrm{Cl}(\psi(A))$. Since $\varphi\psi = \varphi$, we have $\varphi T = \varphi\psi S = \varphi S$ and so φT converges to $\varphi(a)$, hence also the finer net $\varphi T'$ converges to $\varphi(a)$. Since φ is continuous, $\varphi T'$ converges also to $\varphi(b)$. Since $\varphi(K)$ is T_2, we conclude that $\varphi(b) = \varphi(a)$. Hence $b \sim_\psi a$. By $a \in \mathrm{Cl}(A)$, $b \in \mathrm{Cl}(\psi(A))$ and $\mathrm{Cl}(A) \cap \mathrm{Cl}(\psi(A)) = \emptyset$ we conclude $b \neq a$. Hence $b = \psi(a)$, and so $\psi(a) \in \mathrm{Cl}(\psi(A))$. Thus $\psi(\mathrm{Cl}(A)) \subseteq \mathrm{Cl}(\psi(A))$. Applying this relation to $\psi(A)$ and using $\psi^2 = \text{identity}$ we get $\psi(\mathrm{Cl}(\psi(A)) \subseteq \mathrm{Cl}(A)$. Applying ψ again we obtain $\mathrm{Cl}(\psi(A)) \subseteq \psi(\mathrm{Cl}(A))$ and conclude $\mathrm{Cl}(\psi(A)) = \psi(\mathrm{Cl}(A))$.

We show next that if A, B are disjointed closed subsets of K such that $\psi(A) = B$, then $\psi_A = \psi \mid A$ is a homeomorphism of A onto B, thereby establishing (1) $\Rightarrow$ (2). Since ψ_A is one to one and onto B, A is compact and B is T_2, it is enough to show that ψ_A is continuous. Let $C = \varphi(A) = \varphi\psi(A) = \varphi(B)$. Let $\varphi_A = \varphi \mid A$, $\varphi_B = \varphi \mid B$. Since φ_B is a one to one mapping of the compact space B onto the T_2 space C, it is a homeomorphism, and so φ_B^{-1} is a continuous mapping of C onto B. But obviously, $\psi_A = \varphi_B^{-1}\varphi_A$, and so ψ_A is continuous.

(2) $\Rightarrow$ (1). Assume that (1) fails. Then $K/\sim_\psi$ is not normal. Since K is normal, it follows that $\sim_\psi$ is not closed ([3], p. 85, theorem 5). That is, there is a closed

set $A_0 \subseteq K$ such that $A_0 \cup \psi(A_0)$ is not closed. Since A_0 is closed, we conclude that there is a net $S = \langle b_\sigma, \sigma \in \Sigma \rangle$ in $\psi(A_0) \setminus A_0$, converging to b, where $b \notin A_0 \cup \psi(A_0)$. By regularity of K, we may further assume that there is a closed neighborhood V of b such that $a_\sigma \in V$ for all $\sigma \in \Sigma$, and $A_0 \cap V = \emptyset$. Now let $A = \{\psi(b_\sigma) : \sigma \in \Sigma\}$. Then $A \subseteq A_0$ and so $\mathrm{Cl}(A) \subseteq A_0$, while $\psi(A) = \{b_\sigma : \sigma \in \Sigma\} \subseteq V$ and so $\mathrm{Cl}(\psi(A)) \subseteq V$. By $A_0 \cap V = \emptyset$ we have $\mathrm{Cl}(A) \cap \mathrm{Cl}(\psi(A)) = \emptyset$. Now consider $T = \psi S = \langle \psi(b_\sigma), \sigma \in \Sigma \rangle$. This is a net in A, so by compactness there is a finer net T' converging to some $a \in A$. But $\psi T'$ is finer then $\psi T = S$, so it converges to b. By $b \notin A \cup \psi(A)$ we have $\psi(a) \neq b$, and so $\psi \mid \mathrm{Cl}(A)$ is not continuous.

COROLLARY 3.1. *Let K be a compact T_2-space and let A_0, $A_1 \subseteq K$ be disjoint closed subsets of K. Let ψ be a reflection of K such that $\psi \mid A_0$ is a homeomorphism of A_0 onto A_1, and $\psi(c) = c$ for $c \in K \setminus (A_0 \cup A_1)$. Then $K/\sim_\psi$ is T_2.*

§4. Proof of Theorem 3

Let K be an ordered space, ordered by $<$. Let $<'$ be another order on K, and let K' denote K ordered by $<'$. We say that $<'$ *respects* $<$ and that K' is a *respectable reordering* of K iff K' is a reordering of K, and for every k, k is a $<$-semi-isolated point iff k is a $<'$-semi-isolated point.

PROPOSITION 4.0. *Let K be a* SCOS, *and let $A \subseteq K$ be a closed set of semi-isolated points. Then there is a respectable reordering K' of K in which every point of A is upper-isolated.*

Notice that a closed set of upper isolated points in a compact ordered space is well-ordered.

The proof of Proposition 4.0 depends on the following proposition.

PROPOSITION 4.1. *Let $\mathcal{S}$ denote the class of nonzero scattered compact order types. Then $\mathcal{S}$ is the smallest class of order types satisfying:*

(0) $1 \in \mathcal{S}$,

(1) *if ρ is a regular ordinal, and for each $\alpha \leqq \rho$, $s_\alpha \in \mathcal{S}$, then $\Sigma_\alpha^{\rho+1} s_\alpha \in \mathcal{S}$,*

(2) *if $s \in \mathcal{S}$ then $s^* \in S$.*

Obviously, the operations mentioned in (1) and (2) preserve scatteredness and nonzero-compactness. The proof that every nonzero scattered compact ordered type is obtained from the order type 1 by these operations is a straightforward induction on the characteristic and is left to the reader (see e.g. [4]; notice that for $\rho = 1$ (1) means that $\mathcal{S}$ is closed under sum of order types).

PROOF OF PROPOSITION 4.0. The proposition is obvious for a finite SCOS, and clearly holds for K^* whenever it holds for K. Hence, by Proposition 4.1, we conclude by establishing:

CLAIM. *Let ρ be a regular ordinal, and let K_α be a nonempty* SCOS *satisfying Proposition* 4.0 *for every $\alpha \leqq \rho$. Let $K = \Sigma_\alpha^{\rho+1} K_\alpha$. Then Proposition* 4.0 *is true for K.*

PROOF OF CLAIM. Let $A \subseteq K$ be a closed subset of semi-isolated points. Let $A_\alpha = A \cap K_\alpha$. By hypothesis, for every $\alpha < \rho$ there is a respectable reordering K'_α of K_α such that every point in A_α is upper semi-isolated. As in the proof of Lemma 2, $K' = \Sigma_\alpha^{\rho+1} K'_\alpha$ is a respectable reordering of K provided that K_α and K'_α have the same minimal element for every nonisolated $\alpha \leqq \rho$.

Now, if $m_{A_\alpha} = m_{K_\alpha}$ and α is nonisolated, then m_{A_α} is not K-lower-isolated, hence it is K-upper-isolated. Thus, it is isolated in K_α, and so obviously any respectable reordering of K is easily modified to another respectable reordering where m_{K_α} is minimal.

If, on the other hand, $m_{K_\alpha} < m_{A_\alpha}$ let $K_0 = K' + K''$, where $K' = [m_{K_\alpha}, m_{A_\alpha}]$ if m_{A_α} is K-upper-isolated, and $K' = [m_{K_\alpha}, m_{A_\alpha})$ if m_{A_α} is not K-upper-isolated. Since m_{A_α} is K-semi-isolated, K' is a nonempty clopen interval of K_α. Now it is easily checked that if Proposition 4.0 is true of a SCOS, it holds for arbitrary clopen subset (use Proposition 2.0). Hence, K'' has a respectable reordering K''' such that every point of $A \in K''$ becomes upper-isolated. Obviously, $K'_\alpha = K' + K'''$ is then a respectable reordering of K_α where each point of A_α is upper-isolated, and m_{K_α} is minimal. This completes the proof of the claim, and thereby proof of Proposition 4.0.

Let ψ be a reflection of a space K, and let $L = K/\sim_\psi$. L is formally the set of orbits of ψ, with the quotient topology ([3], p. 83). It will be convenient in the sequel to modify the definition and replace an orbit $\{c\}$ consisting of a single point by c.

Let $K = K_0 + K_1$ where K_0 and K_1 are disjoint compact ordered spaces. Let $A_i \subseteq K_i$ be a closed subset of upper-isolated points, and let ψ be a reflection of K such that $\psi \mid A_0$ is a homeomorphism of A_0 onto A_1, and $\psi(c) = c$ for $c \in K \setminus (A_0 \cup A_1)$.

Let $L = K/\sim_\psi$, and let φ be the canonical mapping of K onto L. By our convention, $\varphi(c) = c$ for $c \in K \setminus (A_0 \cup A_1)$.

Let $i \in \{0,1\}$. For $a \in A_i$ let $K_{i,a}$ be the largest K-interval satisfying:

(0) $M_{K_{i,a}} = a$,

(1) $K_{i,a} \cap A_i = \{a\}$.

Since every $a \in A_i$ is upper isolated, $K_{i,a}$ is a nonempty clopen K_i-interval if a is isolated in A_i, and $K_{i,a} = \{a\}$ if a is nonisolated in A_i. Let $K^i = \{k \in K_i : A_i < k\}$. Then K^i is a clopen interval in K_i, and we have:

(2) $K_i = (\Sigma_a^A K_{i,a}) + K^i$.

Set $K_{i,a}^- = K_{i,0} \setminus \{a\}$. For $a \in A_0$ order a subset L_a of L by the requirement

(3) $L_a = K_{0,a}^- + \{\varphi(a)\} + (K_{1,\psi(a)}^-)^*$.

Finally, let L' denote the ordered space obtained from L by the requirement:

(4) $L' = (\Sigma_a^{A_0} L_a) + K^0 + K^1$.

Denote by $<'$ the order of L' and call it the *ψ-order.*

The following properties are clear from the definition:

(5) L_a is a clopen L'-interval whenever $a \in A_0$ is isolated in A_0.

(6) $L_a = \{\varphi(a)\}$ whenever $a \in A_0$ is not isolated in A_0.

(7) K^i is a clopen interval in K_i and in L', inheriting the same order from both spaces.

PROPOSITION 4.2. *Let $K = K_0 + K_1$ where K_0, K_1 are disjoint compact ordered spaces, let $A_i \subseteq K_i$ be closed and let ψ be a reflection of K such that $\psi \mid A_0$ is a homeomorphism onto A_1, and $\psi(c) = c$ for $c \in K \setminus (A_0 \cup A_1)$. Let $L = K/\sim_\psi$ and let L' be the ordered space obtained from L by the ψ-order $<'$ defined above. Then $<'$ is a consistent order on L, and every K-semi-isolated point of $K \setminus (A_0 \cup A_1)$ is also L'-semi-isolated.*

PROOF. The last statement is an immediate corollary of the definitions, so we need only to show that $<'$ is a consistent order on $L = K/\sim_\psi$. By Corollary 3.1, L is T_2. Thus the canonical mapping $\varphi : K \to L$ is closed, and so a quotient mapping. Hence it is enough to show that φ is continuous as a mapping of K onto L'. Since K_i is clopen in K, it is enough to show that $\varphi_i = \varphi \mid K_i$ is continuous, $i = 0, 1$. Since φ_0 is an order preserving mapping of K_0 onto the closed subset $(K_0 \setminus A_0) \cup \varphi(A_0)$ of L', φ_0 is continuous. We show that φ_1 is continuous. Let $K_1' = \Sigma_b^{A_1} K_{1,b}$. Then $K_1 = K_1' + K^1$, and K_1', K^1 are K_1-clopen intervals. Since $\varphi_1 \mid K^1$ is the identity, it is continuous by (7). It is left to show that $\varphi \mid K_1'$ is continuous as a mapping into L'. Since both spaces are ordered spaces, it suffices to prove the following claim.

CLAIM. *Let ρ be a nonzero limit ordinal, and let $\langle c_\alpha, \alpha \in \rho \rangle$ be a sequence in $K_1' = \Sigma_b^{A_1} K_{1,b}$ converging in K_1' to c. Then $\langle \varphi(c_\alpha), \alpha \in \rho \rangle$ converges in L' to $\varphi(c)$.*

Proof of Claim. For $d \in K_1'$ define $\delta(d)$ by the relation $d \in K_{1,\delta(d)}$. Thus, $\delta(d) = \min([d, M_K] \cap A_1)$ and since A_1 is a closed set of upper isolated points, δ is a mapping of K_1' onto A_1. Let $b_\alpha = \delta(c_\alpha)$, and distinguish two cases.

Case 0. *$\langle b_\alpha, \alpha \in \rho \rangle$ is eventually constant.* Let $b \in A_1$, $\gamma \in \rho$ satisfy $b_\alpha = b$ for $\gamma \leqq \alpha < \rho$. Then $c_\alpha \in K_{1,b}$ for $\gamma \leqq \alpha < \rho$.

Case 0.0. *b is isolated in A_1.* Then by (5) $K_{1,b}$ is a clopen interval of K_1, $\varphi \mid K_{1,b}$ is order inverting mapping onto $\{\varphi(\psi(b))\} + (K_{1,b}^-)^*$, which is a closed interval of L'. It is then obvious that $\langle \varphi(c_\alpha), \alpha < \rho \rangle$ converges, to $c = \varphi(c)$ if $c \neq b$, and to $\varphi(\psi(b)) = \varphi(b) = \varphi(c)$ if $c = b$.

Case 0.1. *b is not isolated in A_1.* Then $K_{1,b} = \{b\}$, and so $c_\alpha = b$ and $\varphi(c_\alpha) = \varphi(b)$ for $\gamma \leqq \alpha < \rho$. Thus $b = c$ and $\langle \varphi(c_\alpha), \alpha \in \rho \rangle$ converges to $\varphi(b) = \varphi(c)$.

Case 1. *$\langle b_\alpha, \alpha \in \rho \rangle$ is not eventually constant.* By continuity of the function δ, $\langle b_\alpha, \alpha \in \rho \rangle$ converges to $\delta(c)$. Since A_1 is closed and $b_\alpha \in A_1$ for $\alpha \in \rho$, we conclude that $\delta(c) \in A_1$. Moreover, $\delta(c)$ is a nonisolated point of A_1, since $\langle b_\alpha, \alpha \in \rho \rangle$ is not eventually constant. Hence $K_{1,\delta(c)} = \{\delta(c)\}$. By $c \in K_{1,\delta(c)}$ we have $c = \delta(c)$.

It follows from (2), (3) and (4) that if a is not isolated in A_0, $\langle a_\alpha, \alpha \in \rho \rangle$ is a sequence in A_0 converging to a, and $x_\alpha \in L_{a_\alpha}$ $(\alpha \in \rho)$ then $\langle x_\alpha, \alpha < \rho \rangle$ converges in L' to $\varphi(a)$. Now let $a = \psi(c)$, $a_\alpha = \psi(b_\alpha)$ and $x_\alpha = \varphi(c_\alpha)$. Since $\psi \mid A_1$ is a homeomorphism, $\langle a_\alpha, \alpha \in \rho \rangle$ converges to a, and a is not isolated in A_0. $\varphi(c_\alpha) = x_\alpha \in L_{a_\alpha}$ follows from $c_\alpha \in K_{1,b_\alpha}$, the definition of φ, (3) and $\varphi\psi = \varphi$, as we have:

$$\varphi(K_{1,b_\alpha}) = \{\varphi(b_\alpha)\} \cup K_{1,b_\alpha}^- = \{\varphi(\psi(b_\alpha))\} \cup K_{1,b_\alpha}^- \subset L_{\psi(b_\alpha)} = L_{a_\alpha}.$$

Thus, $\langle \varphi(c_\alpha), \alpha \in \rho \rangle$ converges in L' to $\varphi(a) = \varphi(\psi(c)) = \varphi(c)$.

Proof of Theorem 3. Let K be the topological sum of the respectable scattered compact T_2-spaces K_0 and K_1. Let K' be a T_2-space, and let φ be an order-two mapping of K onto K'. We show that K' has an order that respects φ.

Let $K_i' = \varphi(K_i)$, $i = 0, 1$ and let $A' = K_0' \cap K_1'$. Then K_0', K_1', A' are compact scattered spaces, being closed subsets of K'. Let $\varphi_i = \varphi \mid K_i$, $A_i = \varphi^{-1}(A') \cap K_i$. Then φ_i is an order-two mapping of K_i onto K_i', and since φ is order-two, $\varphi_i \mid A_i$ is a one to one mapping of A_i onto A'. Let $<_i'$ be an order on K_i' that respects φ_i (such an order exists, as K_i is respectable). Since $\varphi_i \mid A_i$ is one to one, each $a \in A'$ is $<_i'$-semi-isolated, and by Proposition 4.0 we may assume that each $a \in A'$ is $<_i'$-upper isolated $(i = 0, 1)$. Let $\tilde{K}_i$ be the set $K_i' \times \{i\}$ ordered by the order $<_i$ defined by $(a, i) <_i (b, i)$ iff $a <_i' b$ $(a, b \in K_i')$. Define $\tilde{\varphi}_i : K_i \to \tilde{K}_i$ by $\tilde{\varphi}_i(k) = (\varphi(k), i)$ $(k \in K_i;\ i = 0, 1)$. Then $\tilde{K}_0$, $\tilde{K}_1$ are disjoint SCOSs, $\tilde{A}_i =$

$A' \times \{i\}$ is a closed subset of upper-isolated points in $\tilde{K}_i$, and $\tilde{\varphi}_i \mid A_i$ is a one to one mapping of A_i onto $\tilde{A}_i$. Let $\tilde{K} = \tilde{K}_0 + \tilde{K}_1$. Define $\tilde{\varphi} : K \to \tilde{K}$ by $\tilde{\varphi} \mid K_i = \tilde{\varphi}_i$, and $\Pi : \tilde{K} \to K'$ by $\Pi((b,i)) = b$ $(b \in K'_i, i = 0,1)$. Since $<'_i$ is a consistent order on K'_i, $\Pi_i = \Pi \mid \tilde{K}_i$ is a homeomorphism of $\tilde{K}_i$ onto K'_i mapping $\tilde{A}_i$ onto A'. Thus, Π, $\tilde{\varphi}$ are order-two mappings, and $\varphi = \Pi\tilde{\varphi}$.

Define a reflection $\tilde{\psi}$ of $\tilde{K}$ by $\tilde{\psi}((a,i)) = (a, 1-i)$ for $a \in A'$, $i = 0,1$, and $\tilde{\psi}((b,i)) = (b,i)$ for $b \in K' \setminus A'$. Then $\tilde{\psi} \mid \tilde{A}_0$ is a homeomorphism of $\tilde{A}_0$ onto $\tilde{A}_1$ and $\tilde{\psi}(c) = c$ for $c \in \tilde{K} \setminus (\tilde{A}_0 \cup \tilde{A}_1)$. Obviously, $\tilde{\psi} = \psi_\Pi$ (see §3).

Let $L = \tilde{K}/\sim_{\tilde{\psi}}$, and let $\tilde{\tilde{\varphi}}$ be the canonical mapping of $\tilde{K}$ onto L. Then we have $\tilde{\psi} = \psi_{\tilde{\tilde{\varphi}}}$. Let $<'$ denote the $\tilde{\psi}$-order on L (see Proposition 4.2) and let L' denote L ordered by $<'$. By Proposition 4.2, $\tilde{\tilde{\varphi}}$ is a mapping of $\tilde{K}$ onto L'. Finally, define mapping Φ of K onto L' by $\Phi = \tilde{\tilde{\varphi}}\tilde{\varphi}$.

Claim. $\varphi(x) = \varphi(y)$ *iff* $\Phi(x) = \Phi(y)$ $(x, y \in K)$.

This follows from $\varphi = \Pi\tilde{\varphi}$, $\Phi = \tilde{\tilde{\varphi}}\tilde{\varphi}$ and $\tilde{\psi} = \psi_\Pi = \psi_{\tilde{\tilde{\varphi}}}$.

Let $\psi = \psi_\varphi$ be the reflection associated with φ. By the claim, $\psi = \psi_\Phi$. Define an ordering $<$ on K' by $\varphi(x) < \varphi(y)$ iff $\Phi(x) <' \Phi(y)$ $(x, y \in K)$. We complete the proof by showing that $<$ respects φ. $<$ is a consistent order on K' since $<'$ is a consistent order on L, by Proposition 4.2. Let $k \in K$ and assume that $\varphi^{-1}(\varphi(k)) = \{k\}$. We show that $\varphi(k)$ is $<$-semi-isolated. By definition of $<$, it is enough to show that $\Phi(k)$ is $<'$-semi-isolated. First note that $k \notin A_0 \cup A_1$, else $\varphi(k) \in A'$ and so $\varphi^{-1}(k) \cap A_0 \neq \emptyset$ and $\varphi^{-1}(k) \cap A_1 \neq \emptyset$, whence $|\varphi^{-1}(\varphi(k))| > 1$. Thus for some $i \in \{0,1\}$ we have $k \in K_i \setminus A_i$, $\varphi(k) \in K'_i \setminus A'$ and $\varphi_i^{-1}(\varphi_i(k)) = \{k\}$. Since $<'_i$ respects φ_i, we see that $(\varphi(k), i) \in \tilde{K}_i$ is $<_i$-semi-isolated, and so by the final clause of Proposition 4.2, $\Phi(k) = \tilde{\tilde{\varphi}}((\varphi(k), i))$ is $<'$-semi-isolated; that is, $\Phi(k)$ is semi-isolated in L'.

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